The Role of Visualization in the Development of Logical-Mathematical and Spatial Thinking

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Abstract

This study addresses the problem of the development of visualization skills in seventh grade elementary school students, which affects the strengthening of logical-mathematical and spatial thinking. The main objective of the research was to strengthen this skill through the implementation of didactic strategies focused on the solution of problems related to spatial and geometric thinking. For this purpose, a qualitative approach was adopted, with a descriptive scope and an action research design. Throughout the intervention, didactic activities focused on improving visual perception and visualization skills of students were designed and applied. The findings showed significant improvements in visualization skills, which resulted in a remarkable progress in the comprehension and resolution of mathematical and spatial problems. The conclusions highlight the importance of integrating visualization strategies in the teaching of mathematics as an effective means for the cognitive development of students, recommending their implementation in curricular programs to promote logical-mathematical and spatial thinking at basic educational levels.

Keywords: Mathematical ability, visualization, perception, geometry.

1. Introduction

The Colombian Ministry of National Education (MEN, 2023), through its Educational System, seeks to strengthen reasoning skills and optimize educational processes to foster the cognitive development of students. This objective is based on a continuous commitment to innovation, academic excellence and constant improvement of educational quality at all levels. Within this framework, the role of the teacher is fundamental, since he/she is responsible for promoting the acquisition and consolidation of cognitive competencies that allow students to process and apply information effectively. This work not only facilitates problem solving and informed decision making, but also enables students to interact effectively and meaningfully in their social and cultural environment.

Sánchez (1995) expresses his concern about the evidence provided by various research studies that show a decrease in the intellectual performance of students. As they advance through the educational levels, an increase in learning difficulties is observed. According to Sanchez, “Many of these difficulties are linked to a lack of information processing skills, which negatively affects the development of schemas that facilitate the storage, retrieval and adequate use of knowledge" (p. 5).

According to Tambychik et al. (2010), students face a combination of difficulties related to mathematical and cognitive skills, which further exacerbates the challenges when solving mathematical problems. These difficulties are likely to result in low performance in this area. Along the same lines, Banhar (2007) argues that "thinking skills are considered part of the processes required in problem-solving efforts" (p. 2). One of the fundamental skills in the field of mathematics, which contributes to overcoming such difficulties, is visualization. This skill is essential for establishing a sound teaching-learning process in secondary education, and classroom work plays a fundamental role in its development.

The ability to visualize offers students multiple opportunities to strengthen both their reasoning and their mathematical practice. Domenicantonio et al. (2011) argue that this ability facilitates the mental formation of images that represent abstract concepts, which is essential to consolidate mathematical learning. Thus, visualization is configured not only as an effective resource for the understanding of various concepts, but also as a fundamental element in the development of mathematical thinking.

Visualization has been consolidated as a fundamental component in mathematical activity, especially perceptible in the geometric field. In this sense, Nardi (2014) highlights the definition proposed by Hershkowitz, who recognizes visualization as one of the languages of mathematics and as one of the various forms of mathematical thinking. For its expression, visual thinking requires a language, whether visual or other, and this language, to be meaningful, must be linked to a conceptual entity.

Educational research has maintained a continuous focus on the development of mathematical skills in basic education, paying special attention to visualization, which is considered fundamental for the teaching and learning of this discipline. This topic has been widely discussed in various congresses and academic events, such as the International Congress of Mathematics Education (ICME 13 and 14), where its relevance has been highlighted through thematic study groups (TSG 20 and TSG 23). Similarly, interest in this dimension is manifested in international forums such as the Congress of the European Society for Research in Mathematics Education (CERME), the Conference of the International Group of Psychology of Mathematics Education (PME), the Latin American Meetings of Educational Mathematics (RELME), the Ibero-American Conference of Mathematics Education (CIAEM) and the Colombian Meetings of Educational Mathematics (ECME). In these spaces, workshops are offered and papers are presented that explore the difficulties, potentials and advances in the teaching of visualization in school contexts.

The relevance of visual aspects in the construction of mathematical knowledge is indisputable. Chalé (2015) argues that visualization plays an important role not only in disciplines traditionally associated with graphical representations, such as geometry, but also in fields that demand rigorous symbolic argumentation, such as algebra.

At CIAEM, Acevedo and Camargo (2015) present some results of a research aimed at taking advantage of visualization processes and skills through a methodological approach that seeks to intuitively approach students to the concepts of rotation and translation.

Researchers such as McGee (1979), Kavale (1982), Presmeg (1986), Del Grande (1990), Zimmermann and Cunningham (1991), Cantoral (2002), De Guzmán (2002), Arcavi (2003) and Acevedo and Camargo (2015) have focused their studies on strengthening visualization skills in the field of mathematics, particularly in the geometric field. These researchers have designed various models, strategies and methodologies, as well as specific activities, aimed at improving mathematical skills. Their focus is especially centered on the use of visualization as a key tool in this process.

In view of the above, this research highlights the skills proposed by Del Grande (1990) to enhance the ability to visualize. In addition, challenging activities are presented for seventh grade that facilitate the development of this skill, emphasizing its importance in the understanding of geometric concepts in the school environment.

2. Literature Review

This section presents a review of the literature relevant to understanding and interpreting the object of study. Key concepts that are fundamental to the analysis and discussion of the topic under study are addressed. These concepts include the conception of visualization, problem-solving theory, and discovery learning. These concepts are essential for a comprehensive understanding of the object of study, as they provide a solid theoretical framework that allows exploration and contextualization of the phenomena under investigation. The review of these concepts allows not only to place the study in an appropriate academic context, but also to identify the interrelationships and practical applications of the theories and approaches discussed.

Visualization

Throughout the history of mathematics, several eminent mathematicians have emphasized visualization as a fundamental element in the mathematical process. De Guzman (2002) indicates that, for the Pythagoreans, visualization was an intrinsic part of mathematical activity. Similarly, Plato emphasizes the fundamental importance of visual representations in mathematical development. Plato argues that, although visual images are often related to graphical representations, they play a crucial evocative role. In his words, “The circle drawn is not reality. The reality is the idea of the circle, but its image plays a very important role as an evocative element of the idea” (p. 4).

Descartes establishes a series of guidelines that integrate visualization processes, highlighting the essential role of images and figures in mathematical reasoning. Likewise, prominent mathematicians of antiquity also recognized the importance of this ability in the advancement of the discipline. De Guzman (2002) notes that Lagrange considered the ability to observe as essential for the mathematician, and Gauss described mathematics as a “science of the eye”.

Accordingly, visualization has been a key skill used by influential mathematicians to advance the development of this science.

The importance of this skill has been highlighted by several studies focused on the teaching and learning of mathematics, with special attention in the field of Mathematics Education.

Different authors have approached visualization in the context of problem solving from different perspectives. Rojas (2009) considers it a heuristic technique, while Cantoral and Montiel (2001) and López (2005) describe it as a crucial skill due to its impact on problem solving and visual-spatial reasoning. In this research, the definition of McGee (1979) is adopted, who describes visualization as that ability that allows the student to mentally manipulate, rotate, rotate and invert objects or figures from some graphic representation. Similarly, Godino et al. (2012) define it as "the ability to evaluate the processes and skills of subjects in performing tasks that require mentally 'seeing' or 'imagining' spatial geometric objects" (p. 2).

Arcavi (2003) highlights the importance of visualization in problem solving, noting that this tool not only facilitates procedures, but can also stimulate innovative and general solutions (p. 224). This perspective coincides with that of Zimmermann and Cunningham (1991), who describe visualization as the ability to create, interpret, use, and reflect on graphs, images, and diagrams. This process not only helps to represent and communicate information, but is also essential for the development of novel ideas and deeper understanding.

In a similar vein, De Guzman (1996) states that:

Mathematical ideas, concepts and methods present a wealth of visual content that can be represented intuitively and geometrically. This representation is highly beneficial both for the presentation and management of these concepts and methods and for their manipulation in the resolution of problems in the field. (p. 2)

Hitt (2002) emphasizes that the visualization process in learning mathematical concepts involves not only observing, but also understanding and manipulating a situation. This process requires the ability to transform a situation from one system of representation to another. The researcher adds:

Mathematical visualization is related to the understanding of a statement and the execution of an activity that, although it does not guarantee the correct answer, may allow the solver to go deeper into the situation being addressed. A key feature of this visualization is the connection between representations, which is essential for the search for solutions to specific problems.(p. 8)

The role of visualization in the teaching and learning of mathematics has undergone a remarkable evolution. Recent studies have proposed a series of models, pedagogical strategies, methodologies and activities aimed at facing and overcoming the difficulties associated with mathematical contents in the educational environment.

The current trend in mathematics teaching is to strengthen the formation and development of skills. Cantoral (2002) recognizes that visualization, as a skill, has often been neglected in teaching, stating that, for students to learn mathematics effectively, it is essential that they are able to visualize concepts. As such, this skill must be cultivated throughout school life, as visualization facilitates mathematical learning. It is essential to design and implement strategies and activities that improve performance in mathematical competencies, as measured by entities such as ICFES and PISA. This requires the integration of visualization in the educational process, as well as the promotion of visual and spatial reasoning skills, especially in the area of problem solving.

De Guzmán (1996) highlights the close relationship between students' visual characteristics and mathematics learning. In this sense, he argues that “since our perception is primarily visual, it is not surprising that visual support is present in mathematical tasks” (p. 2). In line with this perspective, the research focuses on the development of thinking skills, with a particular focus on visualization, following the approaches of Del Grande (1990). See Table 1.

Table 1 Visual Abilities (Del Grande, 1990).

Visual Skills	Description
Eye motor coordination	Ability to align eye movements with body actions.
Visual identification	Ability to distinguish and isolate a specific figure from its surroundings.
Preservation of perception	Ability to understand that an object retains its shape, even if it is not fully visible.
Perception of position in space	Ability to determine the location of an object in relation to oneself or other reference points.
Perception of spatial relationships	Ability to identify and understand the interactions and characteristics between various objects in a spatial environment.
Visual discrimination	Ability to recognize similarities and differences between different objects, images or mental representations.
Visual memory	Ability to recall and visualize the characteristics of objects that are no longer in view.

Note: The table presents the visual skills that students should develop. Source: Suárez and León (2016).

It is essential to consider the contributions of Kulp and Earley (2004) and Kavale (1982) in relation to visual perception, since this is important for visualization by allowing the generation of mental images of abstract concepts and definitions. The development and strengthening of visual skills not only enriches creativity in problem solving, but also affects the formulation of such problems through the mental manipulation of diagrams and representations. These contributions highlight the central importance of visualization skills in the evolution of mathematical thinking.

Problem Solving Theory

Effective mathematical practices must incorporate a number of essential elements to ensure the productivity of teaching interventions. Among these elements are problems, which act as catalysts for inquiry. In this context, Schoenfeld (2020) points out:

A fundamental attribute of a good problem is its ability to foster conjecture, make connections, and allow for abstraction and generalization, as well as the formulation of new problems. The construction of mathematics is based on the constant quest to expand our knowledge through generalizations, abstractions, and the exploration of structural issues. (p. 7)

The teacher's role is to motivate students to face complex problems and to present their solutions clearly and effectively. Therefore, it is relevant that they detail the procedure performed to reach the correct solution, as well as the hypotheses that led to their conclusions. In addition, the teacher should challenge students to develop their creativity, encouraging the use and exploration of various strategies, techniques and approaches to problem solving. It is important to recognize and value those solutions that stand out for their innovation, elegance and significant contribution to the advancement of mathematical thinking.

In accordance with the above Takahashi (2007) states that one of the strategies to provide students not only with knowledge and skills, but also to foster mathematical thinking, is the teaching of mathematics through problem solving. (p. 1)

Different researchers have addressed the definition of the problem, including Pölya (1965), Schoenfeld (1985) (1994); Krulik and Rudnik (1987); Puig (1996), Falk (2001), Pérez (2004), Lesh and Zawojewski (2007), Sriraman and English (2010), Pochulu and Rodríguez (2012) and Santos Trigo (2021).

The activities posed in the classroom constitute significant challenges for students. According to Pérez (2004), challenging problems encourage critical thinking, curiosity, the ability to question, reasoning and the defense of ideas. These problems play a fundamental role in the teaching-learning process, since, by understanding the problem posed, students have the opportunity to approach it in an innovative way and apply various mathematical strategies. Moreover, such challenges not only facilitate learning, but also enrich the educational experience. In this context, Falk (2001) refers that a problem ...

... whose resolution requires the student to construct increasingly complex networks or conceptual maps. This aspect contributes significantly to research in the teaching and learning of mathematics, as well as to the study of the nature and development of mathematical thinking. (p. 10)

In the field of problem solving, this study is based on the phase model outlined by Polya (1965). This model is structured in several fundamental stages: first, the identification and formulation of the problem; second, the design and analysis of the resolution process; third, the implementation of the proposed solution; and finally, a thorough evaluation of both the solution obtained and the procedure employed.

Discovery Learning

By allowing the student to acquire knowledge autonomously through immersion in problematic situations, meaningful and lasting learning is favored. This approach not only stimulates independent thinking, but also encourages the systematic formulation and verification of hypotheses. Consequently, the present research is also based on the discovery learning approach.

Bruner (1984) defines discovery learning as a process in which knowledge is acquired through guided discovery that stimulates curiosity and active exploration. In this approach, the teacher provides appropriate materials that allow the student, through observation, comparison and analysis of situations, to discover autonomously how concepts work. This method not only enhances creative and inferential abilities, but also fosters student autonomy and stimulates their interest in scientific knowledge.

Romero (2011) exposes several principles of discovery learning, highlighting that this type of learning begins with the identification of problems and is developed through an investigative process that facilitates the meaningful resolution of situations. Camargo and Hederich (2010) emphasize that "the discovery learning perspective has generated important and valuable advances in science pedagogy and didactics" (p. 13).

3. Methodology

This study employs a qualitative methodology within the framework of action research. According to Hernández and Mendoza (2018) this qualitative approach seeks to understand phenomena in their natural context, considering the meanings that participants assign to them. This allows describing how they see the world, as well as their beliefs, customs and ways of acting in the face of specific situations. In this context, it is relevant to consider the students' experiences and their contributions as key elements for the analysis.

This approach demands an integral perspective that allows the development of research processes and the consideration of diverse points of view and approaches to a specific problem or situation. In the context of this study, the purpose is to design activities that strengthen visualization skills in mathematics classes. In a qualitative approach, it is paramount that the methodologies, tools and techniques used are aligned with the ontological and epistemological perspective of the researcher. Therefore, an action research design is chosen to guide the application of coherent and effective procedures and strategies to achieve the established objectives.

The action research design is based on Stringer (2007), who points out that:

This approach fosters inclusive participation by allowing the active involvement of all members of a group or community. It is distinguished by its equity, valuing individual contributions and ensuring that the proposed solutions benefit the whole collective. It also has an emancipatory dimension, aimed at combating oppression and social injustice. Its fundamental objective is to improve the living conditions of the participants, promoting their human and personal development.

The instruments for measurement, verification, information gathering and verification were the following: initially, a survey was applied to practicing mathematics teachers, which allowed determining that some of them do not know what this skill consists of, the methodology and the activities that allow strengthening it in seventh grade, a survey validated through the Delphi Method.

A field diary was used to record students' attitudes, skills and knowledge during the classes. In addition, students' productions were collected to provide valuable information that was susceptible to detailed analysis.

During the research, manipulative visual materials, traditional instruments, among others, were used. Through the resolution of challenging problems of geometric content under Wenger's community of practice approach, students learn by themselves, discover and construct knowledge guided by the teacher.

Participants

The participants involved in this research were the active teachers who guide the teaching of mathematics in the seventh grade of elementary education at the Campestre San Diego school in the municipality of Duitama, Boyacá. A total of ten teachers participated with the objective of identifying their knowledge about the visualization skill in the configuration of logical-mathematical and spatial thinking, as well as the activities they implement to strengthen this skill in their students. These teachers were selected due to their direct experience in teaching mathematics and their key role in the development of fundamental competencies in the area, which makes them essential actors to understand the pedagogical strategies applied in the classroom.

On the other hand, 32 seventh grade students also participated, selected through a convenience sampling from the 80 students enrolled in this educational level. The selection of the students was made under previously established criteria, the main one being low school performance in subjects related to logical-mathematical and spatial thinking, a crucial aspect for the focus of this study. Other criteria considered included the willingness to participate in the complementary pedagogical activities designed to improve these skills, regularity in class attendance, which guaranteed that the participants had a minimum of continuity in the educational process, and the evaluation by the teachers, who recommended those students who presented greater difficulties in the comprehension and application of concepts related to the area of mathematics. These criteria made it possible to select a representative group of students with the appropriate profile to analyze the impact of the pedagogical strategies on the development of logical-mathematical and spatial thinking based on visualization.

Phases of the research process

The development of the research was carried out through three defined phases, each of which was designed to address different key aspects of the problem under investigation.

- First phase: Characterization or diagnosis

In this first stage, the main objective was to characterize or diagnose teachers' knowledge and perceptions of visualization as an essential skill for the development of logical-mathematical and spatial thinking. For this purpose, a focused survey was administered to the ten seventh grade mathematics teachers in order to determine their understanding of the role of visualization and its impact on student learning. In addition, we sought to identify the main problems that students face in learning geometry, specifically in relation to the skills necessary for the representation and location of two- and three-dimensional figures in the Cartesian plane, as well as in the interpretation of views of geometric solids. The results of this phase provided a clear vision of the students' difficulties, serving as a basis for the design of the pedagogical intervention.

- Second phase: Design and implementation of pedagogical activities

Based on the emerging results of the first characterization phase, we proceeded to the design and subsequent implementation of a set of pedagogical and didactic activities, focused on problem solving through discovery learning. These activities were specifically designed to strengthen students' visualization skills, facilitating the location and representation of geometric figures, such as polygons and solids, in the Cartesian plane. The objective of this phase was to foster the development of spatial and logical-mathematical thinking through active strategies that promoted interaction and visual understanding of the geometric concepts addressed.

- Third phase: Analysis of the results of the intervention.

The final stage of the research consisted of the evaluation and analysis of the students' opinions and perceptions regarding the pedagogical activities implemented. Through interviews, information was gathered about the students' experience and the impact of the intervention on the development of their visuospatial skills. This phase focused on interpreting the students' arguments and reflections on how the activities contributed to the enhancement of their logical-mathematical and spatial thinking, allowing conclusions to be drawn on the effectiveness of the strategies used and their possible replication in other educational contexts.

4. Results

In this section, the emerging results throughout the three phases of the study are presented. The analysis of the data collected made it possible to identify both the perceptions of teachers and the challenges faced by students in the development of visuospatial skills, essential for the understanding of geometric concepts. The results are organized according to the objectives and phases previously defined in the research process: characterization, design and implementation of pedagogical activities, and analysis of the intervention.

- Results phase one

The findings of the first phase of the research revealed that, in general, seventh grade mathematics teachers at the school lack a structured approach to work on visualization skills in their geometry classes. In the opinion of the informants, most teachers do not employ pedagogical strategies that strengthen the representation and location of two- and three- dimensional figures in the Cartesian plane, thus limiting the development of logical- mathematical and spatial thinking in students. This aspect has generated a significant gap in the learning of essential geometric concepts, since students lack the necessary cognitive tools to solve problems involving spatial visualization. According to several studies (Kozhevnikov et al., 2017; Uttal et al., 2018), visuospatial skills play a fundamental role in learning geometry and their lack of development can negatively affect academic performance in this area.

The informant teachers also point out that one of the limiting factors is the scarce training received in the use of specific methodologies that promote visualization. In the words of the teachers, geometry classes tend to follow traditional approaches, focused on memorizing formulas and procedures, without promoting the mental manipulation of geometric figures. This lack of pedagogical training that encourages visualization is consistent with the observations of Owens and Smith (2018), who emphasize that, without the integration of tools that encourage spatial visualization, students may develop poor geometric thinking. In turn, this prevents students from achieving a deep understanding of mathematical concepts related to spatial geometry, resulting in poor performance in this discipline.

According to the informants, another negative aspect is the lack of adequate didactic materials that favor the development of visualization in the classroom. Teachers have stated that the available technological and manipulative resources are scarce or nonexistent, which hinders the teaching of abstract concepts such as the representation and location of polygons and solids in the Cartesian plane. This situation coincides with what has been argued by Fernández and Hueso (2019), who state that the use of technologies such as dynamic geometry software or interactive applications can significantly enhance the learning of spatial skills. However, the absence of these resources in the school environment remains a major barrier to the development of these skills.

The responses of the reporting teachers have been assigned a unique code. For example, "D1EH" corresponds to Teacher 1 Male Respondent, where:

The letter "D" denotes that it is a Docente.

The number "1" represents the order in which he was surveyed.

The letter "E" indicates that he is a Respondent.

The letter "H" indicates the male gender (Male).

This coding system has been applied uniformly to all participating teachers in order to maintain confidentiality and facilitate reference to their opinions in the analysis of the results.

Some significant responses from teachers regarding the activities they carry out to foster logical-mathematical and spatial thinking are the following:

D1EH: "As a teacher in the area of geometry I have not worked in class on visualization skills because I do not know specific methodologies for this type of teaching. We have been trained in contents, but not in how to teach them."

D8EM: "Students have difficulties visualizing figures in the Cartesian plane, they cannot correctly locate the vertices of polygons and geometric solids and determine their representation according to the views. In my experience, this problem is due to the fact that we do not have manipulative or technological materials that favor this skill."

D4EH: "I have never used software or technological resources to work on visualization in geometry. I think it would be useful, but in our context we do not have access to these tools and we have not been trained in their use either."

D10EM: "I have noticed that most of my students fail to understand how a two- or three-dimensional figure is represented on the plane and how to perform movements on it. This affects their performance on geometry quizzes and tests."

From the analysis of the surveys administered to teachers, it is evident that the lack of specific training and limited access to didactic resources have been determining factors in the lack of pedagogical strategies to develop visualization skills in students. Teachers agree that the traditional approach to teaching geometry does not encourage the development of visuospatial thinking, which significantly limits the understanding and application of geometric concepts. This finding is in line with Lowrie et al. (2017), who emphasize the need to integrate diversified pedagogical approaches that stimulate the mental manipulation of geometric figures from the early stages of learning.

On the other hand, the lack of technological and manipulative resources in the classroom also represents an important obstacle for the strengthening of these skills. Recent studies (Lehrer and Kim, 2020) suggest that the use of interactive software and dynamic geometry tools can have a positive impact on the development of visuospatial skills in students. However, the absence of such resources in the educational context of the participating teachers reflects a structural problem that needs to be addressed at the institutional level.

Globally, in the opinion of the informants, a profound change in geometry teaching methodologies is required, incorporating both teacher training in visualization strategies and the provision of adequate technological resources. If these aspects are not addressed, it is likely that students will continue to face significant difficulties in the representation and location of geometric figures, thus limiting their ability to solve complex mathematical problems. As noted by (Keller and Schreiber, 2019), the development of visuospatial thinking is critical to academic success in mathematics, and its integration in the classroom should be an educational priority.

- Results Phase two

The second phase of the research focused on the creation and implementation of pedagogical activities designed to address the difficulties identified in the diagnostic phase. Based on the previous findings, different didactic strategies were developed to enhance the visualization skills of seventh grade students at Colegio Campestre San Diego de Boyacá, specifically with regard to the representation and location of polygons and solids in the Cartesian plane, their movements and views. The methodology used, based on discovery learning, sought to promote active student participation, facilitating the mental and visual manipulation of geometric figures through problem solving. This approach not only sought to improve geometry performance, but also to strengthen logical-mathematical thinking and spatial understanding, key elements for success in mathematics.

In this sense, the activities focused on the direct interaction of students with geometric concepts, using tools such as graphics, dynamic geometry software and manipulative resources. The implementation of these activities allowed us to evaluate the effectiveness of the proposed pedagogical strategies, observing how students reacted to the new methods and how their visualization skills evolved over the course of the sessions. Below are some problems from Activity 1 used in mathematics class to strengthen students' visualization skills.

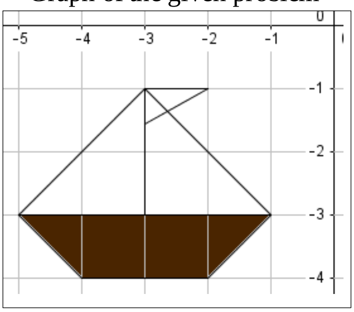
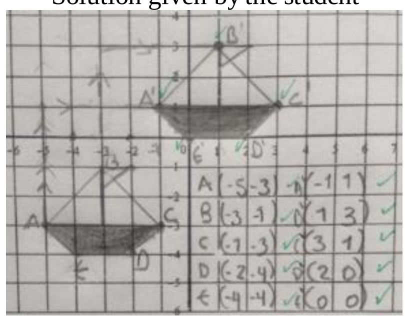
Activity 1: Familiarization with Visualization Skills

Problem 1: The water current.

The purpose of the water current problem is to strengthen skills focused on the perception of position in space and spatial relations, for which the student will have to change positions or represent mirror models.

Santiago has drawn a boat on a Cartesian plane. While playing with this drawing, he invents a story in which the ship, after a strong storm, moves 4 km down and 4 km to the left from its initial location. The picture shows where it is now. If each grid equals one kilometer, make a graph showing where the ship was previously. See Figure 1.

Figure 1 Representation and solution of problem 1.

Graph of the given problem	Solution given by the student
	

Source: Own elaboration.

Some groups had to read the situation several times in order to understand it, given that, in the classes, when dealing with similar problems and exercises, the teacher usually provides the original polygon or figure, and not the one already moved, which generated confusion among the students. After understanding the problem, 80% of the students proposed as a solution to return the ship 4 km upwards and 4 km to the right, thus managing to locate its position in the Cartesian plane before the storm.

Motivated by the situation presented, the students identified the initial and final coordinates of the ship, that is, before and after the storm, although the problem did not specify it. However, it was noticed that some had difficulties in writing the ordered pairs, since they forgot that the first component (x) is located on the abscissa axis and the second (y) on the ordinate axis.

Problem 2: Sandra's journey

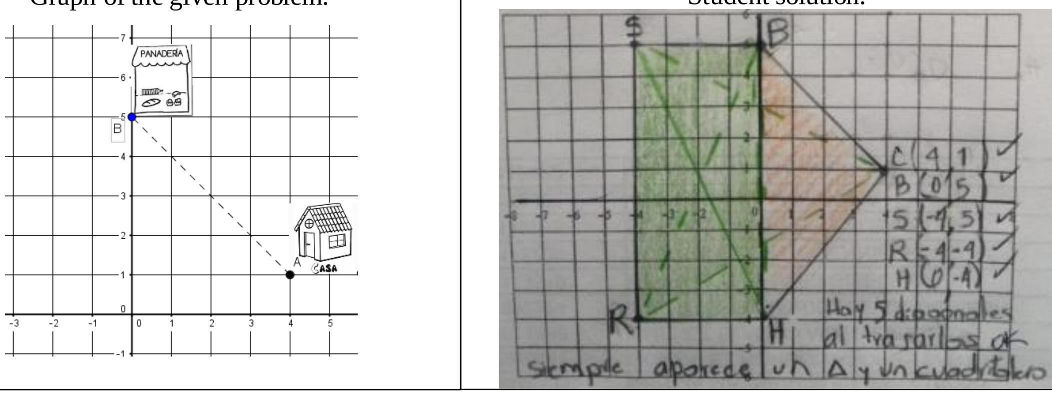
The purpose of the problem was to favor skills aimed at strengthening visual discrimination, as well as the perception of position in space and spatial relationships, which requires relating the position of two or more objects and comparing mathematical objects.

Adriana, Sandra's mother, has asked her to visit certain places to buy some items and has given her the following instructions. When you leave the house (point C); go to the bakery (point B); then to the supermarket, which is 4 km to the left (point S); then go down 9 km to find the stationery store (point P); then go to the ice cream shop, which is at coordinate H (0,-4) and finally return home.

She follows Adriana's directions, remembering that each unit of the grid represents one kilometer. Sandra has noticed that her path is shaped like a pentagon and tells her mother that by walking diagonally from one place to another she could form a quadrilateral and a triangle. Is it possible to verify this statement graphically? When tracing a diagonal, how many different ways of forming a quadrilateral and a triangle result? See Figure 2.

Figure 2 Representation and solution of problem 2

Graph of the given problem.	Student solution.



Source: Own elaboration.

In this problem, 15% of the seventh grade students presented difficulties in interpreting the instructions given about the Cartesian plane, which generated errors in the subsequent questions. In contrast, the other groups were able to complete the irregular pentagon by carefully following the directions.

Regarding the question How many different ways can be created by drawing a diagonal to form a quadrilateral and a triangle, most students identified five different configurations, which they graphed, see Figure 2. They also concluded that by drawing the diagonal in a pentagon only a triangle and a quadrilateral can be formed, since it is not possible to generate other polygons. Some groups supported their answer by analyzing the sum of the internal angles of a pentagon.

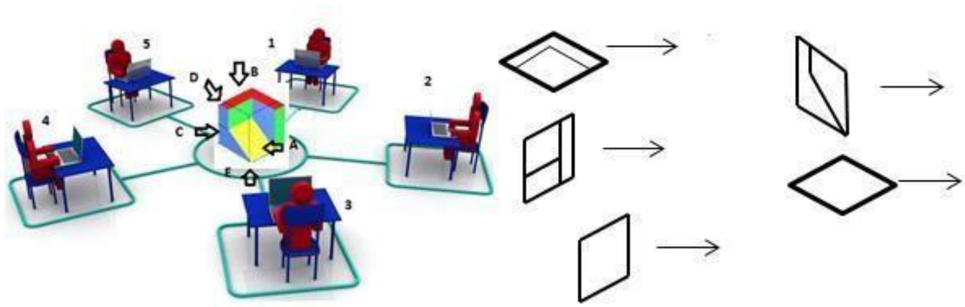
In addition to the above, students were asked to formulate questions related to the problem and then solve them. Some of these were: What is the area of the pentagon? How many diagonals can be drawn on the pentagon? What is the distance between the bakery and the house? among others.

Problem 3: The solid

This situation reinforces skills such as conservation of perception and figure-context, with the objective that the student recognizes the constancy in shape and size of the figure in space.

Graphic designers seek to identify the different views that tourists will have when observing the geometric figure. Each of them will be in charge of the 3D modeling based on the different views of the geometric body. Worker 1 is in charge of the top view (B), 4 of the left side view (C), designer 3 of the front view (A), 2 of the bottom view (E) and the last one of the back view (D). Observe the graph below illustrating this distribution. Classify the views into front, left side, back, top and bottom views. See Figure 3.

Figure 3 Representation of problem 3



Source: Own elaboration

Eighty percent of the students had no difficulty classifying views into front, left lateral, back, top, and bottom; however, they face problems when graphically representing these views in problems.

It is important to note that one of the basic standards of mathematical competencies for seventh grade seeks that students represent three-dimensional objects from different positions and views. Therefore, the proposed activities are oriented to this objective. However, this topic is not addressed in the mathematics classroom due to time limitations and the teacher's underestimation of its relevance for the development of the student's spatial reasoning.

In the socialization of solutions to the previous problems posed, doubts were resolved and errors were identified, while students shared the techniques and tools they used to address the problems to strengthen visualization skills.

Based on the development of the activities, it can be said, globally, that the implementation of these activities focused on learning through discovery, especially in the resolution of geometric problems, has a significant impact on the configuration of logical-mathematical and spatial thinking in students, because during their development, they show interest and willingness to explore different ways of resolution, which allowed them, in turn, to interact and understand more complex geometric concepts. Therefore, the emphasis on visualization helped to develop students' previous knowledge for the location and representation of figures in the Cartesian plane, since in the resolution of problems such as “the water current”, 80% of the participants were able to establish where the boat was before the movement of the storm, which implies a strengthening in spatial perception and the interpretation of translations in the plane.

It was also inferred that the resolution of the problem “Sandra's route” helped to promote visual discrimination and the perception of the spatial relationships of geometric figures. When the students followed the precise instructions and suggestions to follow a path on the plane, most of them correctly identified the position of the places and the configuration of the pentagon; however, some of them presented inconveniences in the position when comparing between two or more objects. From the observation, it was possible to establish that the students configured different possibilities to solve the problem, with different alternatives presenting the correct answers based on the number of combinations to form a triangle and quadrilateral, graphing the number of diagonals that can be traced in the pentagon.

From the observations in the activity “The Solid” it can be inferred that most of the students were able to correctly classify the front, side, top and bottom views of a three-dimensional body. It can be considered that students have an adequate idea of the shape and size of the body in space. However, there are difficulties in the graphic representation of the views. For this reason, it is pertinent to continue working on three-dimensional visualization and spatial reasoning. This shows that some teachers do not work on the topics in detail “because they do not have the time”, which, in the end, prevents students from properly understanding these geometric concepts.

In general, the interaction observed on the socialization of the solutions to the formulated problems created a collaborative environment in which new knowledge was contributed as the reflection on the problems is carried out. Students shared differences in the techniques and tools applied in each activity. In fact, the feedback received also confirmed that the visualization activities not only facilitated the assimilation of geometric concepts, but also promoted logical, critical and spatial thinking in the participants.

Results phase three

In the third phase, students' opinions and perceptions of how the pedagogical activities are developed were analyzed. In the opinions and perceptions, all respondents expressed that geometric concepts could be easily understood through the location of polygons and the movement and interaction in the plane in an easy way through the assigned practical activities. In addition, they stated that, in general, the proposed activities gave them a better understanding of the geometric elements, since interactive and dynamic actions are not implemented in the methods.

Likewise, students who participated in the implementation of the activities reported that it was an enriching and novel experience compared to traditional classes, since for the first time they played an active role in solving problems and constructing their own learning. In addition, the students reported that in particular the problem of "The water current", prompted them to look for multiple ways of how to place the boat in the plane and thus improve not only their perception of movements in space, but also improve their interest in mathematical problems. As for the activity "Sandra's journey", the students indicated that it was challenging but very useful to understand in a clear way how the position of objects gives us the geometric figures that are generated from them.

From the students' perspective, the use of contextualized problem situations and the manipulation of geometric figures in different orientations boosted their ability to reason about space and examine the relative positions between parts. More specifically, during the interviews, several students shared that the graphical representation of solids was one of the most challenging, but, at the same time, meaningful tasks, as it allowed them to visualize how the front, side, and top planes of a three-dimensional figure relate to each other. Although they had problems drawing the different views, students reported that they were able to overcome these limitations with the help of the teacher and group discussion, which allowed them to formulate their statement about solids from this experience.

Students also expressed their opinion about the fact that the socialization of solutions and group discussion before each activity were especially influential aspects in making them feel more confident in tackling mathematical problems. They commented that whether they were forced to share their answers or hear about the various strategies their classmates used opens up a new approach to their own thinking process, allowing them to think of more ways to tackle the same problem. In addition, making use of the opportunity to pose their own questions in relation to the problems allowed them further entrenchment in their logical thinking, as well as realizing more mathematical challenges, such as learning to find the area of some geometric bodies or discovering how many diagonals can be drawn on them.

The analysis of students' perceptions highlights the importance of visualization-oriented activities to enhance the development of logical-spatial thinking. The implementation of contextualized problem situations, as well as the active manipulation of geometric figures, proved to be effective for students to understand abstract concepts and apply them in real scenarios. Through these interactive experiences, there was evidence of a strengthening of their ability to reason about space, interpret relative positions and visualize geometric relationships in different planes. In addition, the focus on graphical representation and socialization of solutions favored the exchange of strategies and promoted a collaborative environment in which students felt more confident in exploring new approaches to solving mathematical problems. In this context, the activities focused on visualization not only facilitated the understanding of geometric elements, but also boosted students' interest in more complex mathematical challenges, consolidating their logical-spatial thinking and their readiness to successfully tackle tasks of greater complexity.

5. Discussion

Based on the emerging results in this research, it can be affirmed that the implementations of pedagogical activities framed in visualization allow the positive development of the students' level of logical-mathematical and spatial thinking. As observed in the analysis of the performance results, the manipulation and representation of geometric figures and objects in experimental and contextualized activities generates an active process of the students in relation to the concepts, which favors their understanding, also ratified by Chavarría and Albanese (2021). This perspective also coincides with the assertion of Cardeñas (2020), Pérez and Fernández (2021) who argue that the visualization of geometric objects from different angles and concrete representation favors the development of spatial reasoning, providing an active and novel framework for the teaching-learning process.

In addition, the perceptions reported by students in this study show that activities such as “The water current” and “Sandra's journey” not only redefined geometric concepts, but also inspired a new approach to drive the exploration of mathematical problems from different points of view. These results are in line with findings by (Jiménez et al., 2022; Martínez and López, 2023) who advocate the importance of facilitating learning opportunities that allow students to construct knowledge through interaction with visual and manipulative representations, as students' positive perception of the freedom to manipulate geometric figures supports the argument that such an approach promotes a higher level of competency in solving complex mathematical problems, and this is relevant in particular for geometry, the central thematic area of this study.

Another relevant aspect of this study was the socialization of strategies and group discussion before and after the activities. According to the results, this proposed approach of this study provided sufficient information to determine that students may feel prepared to overcome the difficulty of graphically representing solids and reasoning through relative spatial positions. This statement is congruent with the thesis of Gonzalez et al., (2019), that both socialization of the approaches combined with exposure to decomposition strategies will not only improve student appreciation, but also allow them to develop more robust mathematical thinking. Observations from the study showed that students take more risks; feel confident and acquire critical thinking as they compare their strategy approaches with methods proposed by other students.

However, it is essential to mention that the incorporation of visualization activities in the present research would not only make the geometric elements in question accessible to the students, but would also encourage them to feel more comfortable approaching situations that involve solving more complex problems. In this case, the phenomenon observed during the interaction could be related to the theory of Lozano and Pérez (2019) who state that “first, developing visualization activities has a considerable impact on the ability to conceptualize. Second, students will be better able to transfer knowledge from one situation to another.” It could be said then that the learning experience analyzed in this study implies the need for further implementation of activities that allow students to “operate” with the properties of figures and, in the good sense of the word, to visualize “transformations” in relation to other, more abstract concepts.

In addition, the results of the experience conducted in this study are consistent with contemporary academic literature. On the one hand, (Rodriguez et al., 2020; Rivera et al., 2022), show the effectiveness of the application of visualization activities in the development of a logical-spatial approach and numerical understanding, and on the other hand, they confirm the possibility for students to master the visualization of the process and manipulation of geometric figures, create interest in performing these tasks and provide conditions for the collaborative process, which facilitates teaching. Therefore, it is advisable to add more regularly to the school curriculum application systems on physical objects and to promote the balanced development of cognitive abilities, which implies mathematical learning that is meaningful and lasting.

6. Conclusions

It can be concluded that visualization is a fundamental aspect in the construction of thinking skills involving logical-mathematical and spatial thinking. In addition, the visualization activity itself not only promotes creativity and exploration of geometric concepts, but also allows the student to apply his strategy in an independent set of problems. Through the analysis of the educational experience, it could be observed that the students not only discovered a much deeper meaning of the construction of the intersection element, but, with a more solid foundation, they were able to construct multiple representations and the same views of geometric bodies.

It is worth mentioning that, despite the initial issues represented by the inability to visualize solids in a plane and abstract objects, the students managed to successfully address this difficulty through interaction and team effort. Communication of their approaches and discussion of ideas helped to integrate a more structured type of thinking and resulted in greater coherence. This point emphasizes the importance of advocating activity that stimulates dialogue and joint reflection to build knowledge in mathematics.

On the other hand, it is concluded that activity-based visualization should be incorporated as part of the mathematics curriculum, since it helps students to improve their ability to interpret geometric figures and spatial perception. With these tasks, students are given the opportunity to manipulate figures and perform transformations in the plane and, which allow them to identify geometric relationships in a much more intuitive way. This in turn provides them with a stronger foundation for tackling more difficult problems.

Therefore, it can also be inferred that the development of key figure-context skills and perception of position in space create the proper foundation for a solid understanding of geometric concepts. In general, activities that include the presentation of two- and three-dimensional objects are highly engaging for students, increasing their level of interest and enthusiasm and helping them to approach the problem situation with a more positive attitude.

Finally, it is concluded that it is essential to continue designing and implementing activities with contextualized and challenging problems, where students can ask questions at their discretion and offer argued solutions. These initiatives not only facilitate the development of the corresponding visual and spatial skills, but also contribute to a collaborative and dynamic learning environment; that is, education as experienced and explored knowledge for critical thinking.

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